Journal of Statistical Physics, Vol. 126, No. 2, January 2007 (© 2006)
DOI: 10.1007/s10955-006-9259-8

A Statistical Mechanics Approach for a Rigidity Problem

Alejandro Mesón[1] and Fernando Vericat[1]

Received August 23, 2006; accepted November 14, 2006
Published Online: January 5, 2007

We focus the problem of establishing when a statistical mechanics system is determined by its free energy. A lattice system, modelled by a directed and weighted graph $\mathcal{G}$ (whose vertices are the spins and its adjacency matrix M will be given by the system transition rules), is considered. For a matrix $A(q)$, depending on the system interactions, with entries which are in the ring $\mathbf{Z}[a^q : a \in \mathbf{R}^+]$ and such that $A(0)$ equals the integral matrix M, the system free energy $\beta_A(q)$ will be defined as the spectral radius of $A(q)$. This kind of free energy will be related with that normally introduced in Statistical Mechanics as proportional to the logarithm of the partition function. Then we analyze under what conditions the following statement could be valid: if two systems have respectively matrices A, B and $\beta_A = \beta_B$ then the matrices are equivalent in some sense. Issues of this nature receive the name of rigidity problems. Our scheme, for finite interactions, closely follows that developed, within a dynamical context, by Pollicott and Weiss but now emphasizing their statistical mechanics aspects and including a classification for Gibbs states associated to matrices $A(q)$. Since this procedure is not applicable for infinite range interactions, we discuss a way to obtain also some rigidity results for long range potentials.

KEY WORDS: rigidity problems, statistical mechanics systems, free energy, Gibbs states

1. INTRODUCTION

Let Ω be a finite set of spins and M a positive integral $card\,\Omega \times card\,\Omega$–matrix with entries $M(i, j)$. The space of admissible configurations is defined as the set $\Sigma_M = \{x : x = (x_k)_{k\in\mathbf{Z}^+} : M(x_k, x_{k+1}) \neq 0, x_k \in \Omega\}$ according to which any site k has a spin $x_k \in \Omega$. The matrix M can be interpreted as giving the transition

[1] Instituto de Física de Líquidos y Sistemas Biológicos (IFLYSIB), CONICET–UNLP-CICPBA and Grupo de Aplicaciones Matemáticas y Estadísticas de la Facultad de Ingeniería (GAMEFI) UNLP, La Plata, Argentina; e-mail: meson@iflysib.unlp.edu.ar; vericat@iflysib.unlp.edu.ar.

rules for the interactions between sites. From M a finite directed graph $\mathcal{G}_M$ can be constructed whose vertices are labeled by the elements of Ω and where there are $M(i, j)$ edges from a vertex i to a vertex j.

Let $Exp = \{a^q : a \in \mathbf{R}^{++}\}$, where $\mathbf{R}^{++} = \{a \in \mathbf{R} : a > 0\}$, thus the ring $\mathbf{Z}[Exp]$ will be formed by the maps $f : \mathbf{R} \to \mathbf{R}^{++}$ with $f(q) = \sum_{i=1}^{r} n_i a_i^q, n_i \in \mathbf{Z}$. Similarly is considered the ring $\mathbf{Z}^+[Exp] = \{f : f(q) = \sum_{\ell=1}^{r} n_\ell a_\ell^q, n_\ell \in \mathbf{Z}^+\}$.

The set of $m \times m-$matrices $A(q)'s$ with entries in $\mathbf{Z}^+[Exp]$ is denoted by $\mathcal{M}_m(\mathbf{Z}^+[Exp])$. Then we associate to each $A(q) \in \mathcal{M}_m(\mathbf{Z}^+[Exp])$ the directed graph: $\mathcal{G}_{A(q)} := \mathcal{G}_{A(0)}$. Notice that $A(0)$ is an integral matrix. For any pair of vertices of $\mathcal{G}_{A(q)}$ the entry i, j of $A(q)$ has the form $A(q)(i, j) = \sum_{\ell=1}^{r} a_\ell^q(i, j)$, so that there are exactly $r = r(i, j)$ edges from the vertex i to the vertex j. The coefficients a_ℓ are now bijectively assigned to these edges.

Thus we may have determined a one-dimensional statistical mechanics system by $A(q)$ and the consequents $\mathcal{G}_{A(q)} := \mathcal{G}_{A(0)}$, $\Sigma_{A(q)} := \Sigma_{A(0)}$. For instance in the basic and well known Ising model $r = 1$, $a_\ell(i, j) = \exp(\mathcal{J}ij)$, $i, j \in \{-1, 1\}$ ($\mathcal{J}$ is the coupling parameter). For simplicity, we shall denote the associated graph directly by $\mathcal{G}_A$ and the space of configurations by Σ_A. For a configuration $x \in \Sigma_A$, we denote by $\Pi_n(x)$ the truncation of the infinite sequence x to its first n terms, therefore $\Pi_n(x)$ will be represented by a path γ in $\mathcal{G}_A$. Each path will be a sequence $\gamma = e_0 \ldots e_{n-1}$, where each e_i is an edge in $\mathcal{G}_A$ from one vertex to another. In this case we shall say that the path has length n and we write $|\gamma| = n$. Now, we can consider the space of configuration Σ_A constituted by "infinite length" paths $\gamma = e_0 e_1 \ldots$, with e_i is an edge in $\mathcal{G}_A$, and where both initial vertex and terminal vertex of e_i are identified with spins of the system. We call admissible paths in $\mathcal{G}_A$ to those representing admissible sequences. To any edge e of the graph $\mathcal{G}_A$ will be assigned a weight $w_A(e)$ and the path $\gamma = e_1 \ldots e_n$ will have weight $w_A(\gamma) = \prod_{i=0}^{n-1} w_A(e_i)$.The closed paths in $\mathcal{G}_A$ are called *cycles*. For instance, for Markov systems the weight assigned to any edge from a vertex i to a vertex j is a probability $P_{i,j}$. This example shows that the consideration of weighted graphs allows to study more general systems than particles interacting via pair wise potentials, like the above mentioned Ising model.

For any spin $i \in \Omega$, let $e_{i,0}, \ldots, e_{i,r-1}$ the edges in $\mathcal{G}_A$ starting in i, thus for any pair $(i, e_{i,j})$ we can form a vector v_i indexed by the vertices of $\mathcal{G}_A$, which jth-coordinate will be given by $w_A(i, e_{i,j})$, where $w_A(i, e_{i,j})$ is the weight of the edge from i to j. By the bijective assignation of the coefficients to the edges the entries of the matrix $A(q)$ are given by weights of the edges, in particular the sum $\sum_{j=1}^{r} w_A(i, e_{i,j})$ equals the ith-row of $A(q)$.

For a system $(\mathcal{G}_A, \Sigma_A)$, where A is an irreducible matrix over $\mathbf{Z}^+[\exp]$ we introduce the *free energy*

$$\beta_A(q) = \rho(A(q)), \tag{1}$$

where $\rho(A)$ is the spectral radius of A. By the Perron–Frobenius theorem $\beta_A(1)$ has a single eigenvalue.

We shall also consider a definition of a *free energy function per particle* in the more customary way from the partition function $Z_n(q) = Z_n(q, A) = \sum_{|\gamma|=n} w_A^q(\gamma)$, where the sum is extended over all the cycles of length n:

$$F_A(q) = \lim_{n\to\infty} \frac{1}{n} \log(Z_n(q)). \tag{2}$$

Thus q can be interpreted as the inverse of the temperature.

The quantities $\beta_A(q)$ and $F_A(q)$ can be related in the following way: a string $(i_0, i_{1,\ldots,} i_{n-1})$ is called admissible for a positive integral matrix A if $A(i_0, i_1) \times A(i_1, i_2) \times \cdots \times A(i_{n-2}, i_{n-1}) \neq 0$. A $n-$periodic string is that in which $i_{n-1} = i_0$, therefore there is a one to one correspondence between the periodic strings and the cycles of the graph associated to A. In this way if P_n denotes the set of the set of $n-$ periodic strings then $card\, P_n = \sum_{(i_0, i_{1,\ldots,} i_{n-1})} A(i_0, i_1) \times A(i_1, i_2) \times \cdots \times A(i_{n-2}, i_{n-1}) = Tr(A^n)$ and so that

$$Z_n(q) = Tr(A^n(q)), \tag{3}$$

from which is obtained $\beta_A(q) = \exp(F_A(q))$.

Another important quantity is the Ruelle zeta function[11]

$$\zeta_A(z, q) = \exp\left[\sum_{n=1}^{\infty} Z_n(q) \frac{z^n}{n}\right], \tag{4}$$

which gives an analytical map in the disc $|z| < \exp(\beta_A(q))$. We can express the zeta map as $\zeta_A(z, q) = \exp[\sum_{n=1}^{\infty} Tr(A^n(q))\frac{z^n}{n}] = \exp[Tr(-\log(I - zA(q)))] = \frac{1}{\det(I - zA(q))}$.

To any system $(\mathcal{G}_A, \Sigma_A)$ can be associated the *non-marked spectrum* $\mathcal{S}_A = \{(w_A(\gamma), n) : \gamma \text{ is a cycle with } |\gamma| = n\}$ and the *marked spectrum* $\mathcal{L}_A = \{(w_A(\gamma), \gamma) : \gamma \text{ is a cycle with } |\gamma| = n\}$. There are different equivalence relations between matrices in such a way that elements in the same equivalence classes share the same spectrum. If $A, B \in \mathcal{M}_n(\mathbf{Z}^+[\exp])$, let

i) $A \sim_1 B$ if and only if $\log w_A(e_0) = \log w_B(e_0) + U(e_0) + U(e_1)$ for any edge with initial vertices e_0, e_1 and for some map U.
ii) $A \sim_2 B$ if and only if $A(q)(i, j) = B(q)(\sigma(i), \sigma(j))$, for any i, j, where $\sigma : \Omega \to \Omega$ is some permutation of the states.

Then is valid: $A \sim_1 B \Rightarrow \mathcal{L}_A = \mathcal{L}_B$ and $A \sim_2 B \Rightarrow \mathcal{S}_A = \mathcal{S}_B$, and on the other hand $\mathcal{S}_A = \mathcal{S}_B \Rightarrow \beta_A = \beta_B$. One of the objectives of this article is to analyze under which conditions the free energy determines the matrix or the spectrum of the system, up to equivalence, i.e. when, in some sense, the reciprocal of the above implications hold, or in other words when the free energy is a complete invariant. This falls in the category of the so called *rigidity problems*.

Remark. To obtain equivalent matrices necessarily the corresponding graphs should be isomorphic. Recall that two graphs $\mathcal{G}_1, \mathcal{G}_2$ if there is one to one map φ which carries any vertex of $\mathcal{G}_1$ in a vertex of $\mathcal{G}_2$ and such that if there are k edges from v_1 to v_2 then there are k edges from $\varphi(v_1)$ to $\varphi(v_2)$. Now to produce examples of non-equivalent matrices it must be considered non isomorphic graphs.

Another issue to be considered is about the Gibbs states: if x is a configuration and $\gamma = e_0 \dots e_{n-1}$ is the path in $\mathcal{G}_A$ obtained from the truncation of x to the first n symbols then $H_n(\gamma) := -\log w_A(\gamma) = -\sum_{i=0}^{n-1} \log w_A(e_i)$ may be seen as describing the interaction between the spins in $\mathcal{G}_A$. The interaction on the entire configuration can be written as $H_n(\gamma) + W(\gamma \mid \gamma^c)$, where $W(\gamma \mid \gamma^c)$ describes the interaction energy between the spins joined by the edges in γ and those joining the remaining spins of the configuration. The choice of paths in $\mathcal{G}_A$ correspond to a selection of certain boundary conditions for the spin system, if periodic boundary conditions are chosen then the cycles are considered. Thus for the Hamiltionian H_n the *Gibbs ensemble of finite volume n* can be taken as the probability measure

$$\mu_{n,A(q)}(\{\gamma\}) = \frac{w_A^q(\gamma)}{\sum_{|\gamma|=n} w_A^q(\gamma)} = \frac{w_A^q(\gamma)}{Z_n(q)} = \frac{\exp(-qH_n)}{Z_n(q)}, \quad \gamma = e_0 \dots e_{n-1}.$$

The *Gibbs states* $\mu_{A(q)}$ associated to a matrix $A(q)$ are weak accumulation points of finite volume ensembles, i.e.

$$\mu_{A(q)} = \lim_{k\to\infty} \mu_{n_k, A(q)},$$

for some sequence $\{n_k\}$. By the compactness of the space of measures on Σ_A, such accumulation point does exist.

For $\gamma = e_0 \dots e_{n-1}$ the *cylinder* $C_n = C_n(\gamma)$ is the set

$$\{x \in \Sigma_A : x_i = e_i, i = 0, 1, \dots, n-1\}.$$

We shall prove that for every n, q and for any path γ of length n, there are constants $A_1, A_2 > 0$ such that

$$A_1 \le \frac{\mu_{A(q)}(C_n(\gamma))}{w_A^q(\gamma)\exp(-nF_A(q))} \le A_2 \tag{5}$$

or

$$A_1 \leq \frac{\mu_{A(q)}(C_n(\gamma))}{w_A^q(\gamma)(\beta_A(q))^{-n}} \leq A_2. \tag{6}$$

So that these Gibbs states become equilibrium states, for the free energies F_A or β_A, in the sense of the Ruelle thermodynamic formalism.[11] For irreducible matrices in $\mathcal{M}_m(\mathbf{Z}^+[\exp])$ the map $\beta_A(q)$ is real analytic[12] and so by the thermodynamic formalism there is an unique Gibbs state $\mu_{A(q)}$ for each real q. Recall that a matrix A is irreducible if for any i, j there is a positive integer m such that all the entries of $H^m_{i,j}$ are strictly positive. A matrix A is *aperiodic or transitive* if there is a positive integer m such that all the entries of $H^m_{i,j}$ are strictly positive.

We shall obtain a classification of the equilibrium states in terms of the unmarked spectrum, more specifically the result to be presented reads: $\mu_A = \mu_B$ if and only if there is a constant $C > 0$ such that $w_A(\gamma) = w_B(\gamma)C^n$, for any positive integer n and for any cycle γ with $|\gamma| = n$.

In Ref. 10, Pollicott and Weiss have considered a free energy obtained from a partition function defined as the statistical sum of the potential over the periodic points of the dynamic map. They proved that for finite range potentials this free energy determines the potential up to some equivalence. The free energy they consider is associated to finite range potentials $\varphi : \Sigma_A \to \mathbf{R}$, depending on a finite number of coordinates. For example depending on two coordinates, the matrix $A(q)$ can be defined by $A(q)(i, j) = A(i, j)\exp(q\varphi(x_0, x_1))$, with $x_0 = i$, $x_1 = j$. Although some of our proofs for the finite interaction case closely follows those from Pollicot and Weiss, our framework (using directed graphs) is more general in the sense that it is valid not only for interaction potentials, but for more general situations, e.g. Markov chains.

If we are in the more general situation in which potentials depend on the whole configuration, we must work with other class of objects than matrices. They will be *transfer operators,* in the style of those introduced by Ruelle in his thermodynamic formalism, and the aim will be to obtain some kind of rigidity result.

2. COMPLETE INVARIANCE OF THE FREE ENERGY FOR FINITE INTERACTIONS

Let us consider the polynomial $D_A(z, q) = \det(I - zA(q)) \in \mathcal{R}[z]$ ($\mathcal{R} = \mathbf{Z}^+[\exp]$), so that $D_A(1/\beta_A(q), q) = 0$. By a result in Ref. 12 it can be established that $D_A(z, q)$ is minimal for β_A in the following sense: if $Q \in \mathcal{R}[z]$ is another polynomial with $1/\beta_A(q)$ as a root, with A an irreducible matrix, then D_A divides Q in $\mathcal{R}[z]$. There is a direct relationship between D_A and the characteristic polynomial $P_A(z, q) = \det(zI - A(q))$ of the matrix A, indeed $D_A(z, q) = z^m P_A(z^{-1}, q)$, for a $m \times m-$matrix. Therefore the characteristic polynomial is also minimal among those for which $\beta_A(q)$ is a zero. If it were proved that D_A is irreducible then $\beta_A(q)$ would determine D_A, because the polynomial for the free energy is minimal. The same occurs for the characteristic polynomial by the relationship of above.

Proposition 1. *Let $A = A(q)$ be an aperiodic matrix with entries in $\mathcal{M}_m$ ($\mathbf{Z}^+[\exp]$) having the following property: any non-trivial product of powers of its entries is different form the unity. Then $D_A(z, q)$ is irreducible, i.e. it cannot be written as a product of two non-constant polynomials in $\mathcal{R}[z]$.*

Proof: We can express $D_A(z, q) = \det(I - zA(q)) = 1 + \sum_{\ell=1}^{m} C_\ell(q) z^\ell$, where

$$C_\ell(q) = \sum_{\substack{(i_1, i_2, \ldots, i_r) \\ i_1 + i_2 + \cdots + i_r = \ell}} \frac{(-1)^r}{r!} \prod_{j=1}^{r} \frac{1}{i_j} tr(A^{i_j}(q))$$

or also $D_A(z, q) = \prod_{i=1}^{m}(1 - zE_i)$, where $E_i = E_i(q)$ are the eingenvalues of A, counted with their algebraic multiplicity. For instance for $m = 2$ we have with the first expression $D_A(z, q) = z^2 - tr(A(q))z + [tr(A(q))^2 - tr(A^2(q))]z^2 = z^2 - tr(A(q))z + \det(A(q))$ and with the second one $D_A(z, q) = z^2 - (E_1 + E_2)z + E_1 E_2$, of course both two expressions are equal by the invariance of the of conjugation A with a diagonal matrix. We can thus interchange the coefficients, for instance we may adopt the development $1 - tr(A(q))z + \cdots + (\prod_{i=1}^{m} E_i)z^m$.

Let us assume that there exists $R(z, q)$, $S(z, q) \in \mathcal{R}[z]$, non-constant, such that $D_A(z, q) = R(z, q). S(z, q)$ or

$$1 - tr(A)z + \cdots + \left(\prod_{i=1}^{m} E_i\right) z^m = (1 + R_1 z + \cdots + R_{m-k} z^{m-k}) \times (1 + S_1 z + \cdots + S_k z^k), \tag{7}$$

with $R_{m-k} = \sum_i n_i e_i^q$ and $S_k = \sum_j m_j f_j^q$. By comparing the terms of D_A and the product of R and S, we firstly have $\sum_{i,j} n_i m_j e_i^q f_j^q = \pm \prod_{i=1}^{m} E_i$, so that the $n_i m_j$ will be equal to a product $\prod_j A(j, \sigma(j))$, for some permutation σ of n elements. If are now compared the coefficients of z^{m-k} and z^k deduces that the e_i^q and f_j^q have the form $\frac{\prod_j A(j,\sigma(j))}{\prod_{\ell=1}^{m=k} A(i_\ell, i_\ell)}$, where σ is a permutation which in one case fixes the indexes $(i_1, \ldots, i_k)$ and the $(i_1, \ldots, i_{m-k})$ in the other. However the term $\prod_{i=1}^{n} A(i, \sigma(i))$, with σ a permutation with no fixed point, will appear in some term of $\det(I - zA(q)$, but it cannot be expressed as a product $e_i^q f_j^q$ by the enunciated property of the matrix. Let us display this situation for $n = 2$, the coefficient of z^2 contains sums of elements of the form $A(i, i)A(j, j) - A(i, j)A(j, i)$ and it should be needed $A(1, 2)A(2, 3)A(3, 4)A(4, 1)$, which corresponds to the cyclic permutation

$(1, 2, 3, 4) \to (2, 3, 4, 1)$, equal a term of the form $A(i, j)A(j, i)A(r, s)A(s, r)$, which is not possible by the property of the matrix. $\square$

We illustrate with two examples:

Example 1. We consider the Ising model, for which $A(q) = \begin{pmatrix} \exp(\mathcal{J}q) & \exp(-\mathcal{J}q) \\ \exp(-\mathcal{J}q) & \exp(\mathcal{J}q) \end{pmatrix}$. If $D_A(z, q)$ could be factorized as a product of two linear non-constant polynomials then it would have: $D_A(z, q) = \det(I - zA(q)) = z^2 - tr(A(q)z + \det(A(q) = (z - \sum_i n_i e_i^q)(z - \sum_j m_j f_{ji}^q)$, and so $z^2 - tr(A(q)z - \det(A(q) = z^2 - (\sum_i n_i e_i^q + \sum_j m_j f_{ji}^q)z + \sum_{i,j} n_i m_j e_i^q f_j^q$. From this is obtained that $\sum_i n_i e_i^q + \sum_j m_j f_j^q = tr(A(q) = 2\exp(\mathcal{J}q)$ and $\sum_{i,j} n_i m_j e_i^q f_j^q = \det(A(q) = \exp(2\mathcal{J}q) - \exp(-2\mathcal{J}q)$. Therefore $n_i m_j = \exp(2\mathcal{J})$ or $n_i m_j = \exp(-2\mathcal{J})$, in particular $\exp(4\mathcal{J}) = 1$, but it is no possible unless $\mathcal{J} = 0$, which is not the case.

Example 2. Let us consider a more bit general interaction, whose matrix has entries $A(q)(i, j) = \exp(-qa(i, j))$, $i, j = 1, 2$, for a given $a > 0$ depending of two spins. Now $D_A(z, q) = z^2 - tr(A(q))z + \det(A(q))$ can be factorized in $\mathcal{R}[z]$ as product of two linear factors whenever the discriminant $\Delta = (tr(A(q))^2 - 4\det(A(q))$ could be expressed as $(\sum_i n_i e_i^q)^2$, for some n_i, e_i.We have $tr(A(q) = \exp(-qa(1, 1)) + \exp(-qa(2, 2))$ and $\det(A(q)) = \exp(-qa(1, 1))\exp(-qa(2, 2)) - \exp(-qa(1, 2))\exp(-qa(2, 1))$. Thus $\Delta = [\exp(-qa(1, 1)) - \exp(-qa(2, 2))]^2 - 4\exp(-qa(1, 2))\exp(-qa(2, 1))$, and hence the condition on the discriminant cannot be satisfied since $\exp(-qa(1, 2))$ $\exp(-qa(2, 1)) \neq 0$.

To see how the free energy determines $D_A(z, q)$ in the case of 2×2 matrices, let us notice that $\frac{1}{\beta_A(q)} = \frac{tr(A(q) + \sqrt{(tr(A(q))^2 - 4\det(A(q))}}{2}$, therefore if $\Delta = (tr(A(q))^2 - 4\det(A(q) \neq 0$ then the free energy determines $tr(A(q)$ as well as $\sqrt{(tr(A(q))^2 - 4\det(A(q))}$ and so $\det(A(q))$ will be also determined by $\beta_A(q)$. Now $D_A(z, q) = z^2 - tr(A(q)z + \det(A(q))$ is completely determined by the free energy.

Thus we have, as by the comment of above, that the free energy determines the minimal polynomial $D_A(z, q)$ and also the characteristic polynomial of A.

Lemma 1. *The zeta function $\zeta_A(z, q)$ determines the spectrum $\mathcal{S}_A$.*

Proof: Let $\zeta_A(z, q) = \exp[\sum_{n=1}^{\infty} Z_n(q)\frac{z^n}{n}]$, which has radius of convergence $\exp(\beta_A(q))$. Now the coefficients of the series can be uniquely obtained deriving with respect to q. Then from $Z_n(q) = \sum_{|\gamma|=n} w_A^q(\gamma)$ the numbers $w_A(\gamma)$ are uniquely determined up to permutation, this follows from Newton identities. So the spectrum is completely determined from the zeta function. □

Therefore combining Proposition 1 and Lemma 1, and recalling that $\zeta_A(z,q) = \frac{1}{\det(I - zA(q))} = \frac{1}{D_A(z,q)}$, we have that for matrices, with the property in the statement of the Proposition1 the free energy $\beta_A(q)$ determines the spectrum $\mathcal{S}_A$, or $\beta_A = \beta_B \Longrightarrow \mathcal{S}_A = \mathcal{S}_B$.

Proposition 2. *The polynomial $D_A(z,q)$, with A an aperiodic matrix with entries in $\mathcal{M}_n(\mathbf{Z}^+[\exp])$, determines the $\sim_2$ equivalence class of the matrix A, or $D_A = D_B \Longrightarrow A \sim_2 B$.*

Proof: Let us consider the development of $D_A(z,q)$ displayed in the above proposition (the first one):

$$\det(I - zA(q)) = 1 + \sum_{i=1}^{n} C_i(q) z^i, \text{ with}$$

$$C_n(q) = \sum_{\substack{(i_1, i_2, \ldots, i_r) \\ i_1 + i_2 + \cdots + i_r = n}} \frac{(-1)^r}{r!} \prod_{j=1}^{r} \frac{1}{i_j} tr(A^{i_j}(q)).$$

The terms in $tr(A)$ can be determined by the Lemma 1. The coefficient of z^2 consists of elements with the form $A(q)(i,i)A(q)(j,j) - A(q)(i,j)A(q)(j,i)$, by the invariance of $\det(I - zA(q))$ by conjugation by diagonal matrices it can be considered without loss of generality that $A(j,1) = 1$, for any j. Then the products $A(q)(i,j)A(q)(j,i)$ can be calculated from the elements $A(q)(i,i)A(q)(j,j)$ belonging to the trace and so possible of be computed from the earlier step.

The coefficient of z^3 involves triple products of the form $A(q)(i,i)$ $A(q)(j,j)A(q)(k,k)$ and triple products of entries of $A(q)$ with different coordinates. The terms with the same coordinate are known. In particular in the expression appear terms $A(q)(i,i)A(q)(j,k)A(q)(k,i)$, the $A(q)(i,i)$ as we say are already determined and by the above process can be also obtained, and also by above can be determined elements of the form $A(q)(i,i)A(q)(j,k)A(q)(k,j)$. To obtain the general term $A(q)(k,i)$, let

$$A(q)(1,i)A(q)(i,j) = \frac{A(q)(1,i)A(q)(i,j)A(q)(k,i)}{A(q)(k,i)}, \tag{8}$$

the element $A(q)(1,i)A(q)(i,j)$ is a factor in a term of the coefficient of z^3 and the $A(q)(1,i)$ are already known. In the coefficients of z^3 appear products of the form $A(q)(i,j)A(q)(j,k)A(q)(k,i)$ and in particular those of the form $A(q)(i,1)A(q)(1,k)A(q)(k,i) = A(q)(1,k)A(q)(k,i)$, then multiplying (10) by $A(q)(1,k)$ is obtained a term in the the coefficient of z^3, and thus is determined

$A(q)(k, i)$ by known entries. This process can be inductively iterated to recover the coefficients of $D_A(z, q)$.

The characteristic polynomial of a matrix A is invariant by conjugation of A by a permutation matrix, i.e. a matrix in which any column, or row, is a vector with only coordinate equal 1 and the others 0. For a $m \times m$–matrix there are $m!$ permutation matrices and so it can be proved that there are many finitely matrices with the same characteristic polynomial. Therefore the characteristic polynomial, and do the polynomial D_A determines the matrix, up to to conjugation by permutation matrices.

Let now $A(q)$, $B(q)$ such that $B(q) = P^{-1}A(q)P$, where P is a permutation matrix. If σ is a permutation of m elements, i.e. $\sigma : \{1, 2, \ldots m\} \to \{1, 2, \ldots m\}$ bijective, then we denote $(B \circ \sigma)(q)(i, j) = B(q)(\sigma(i), \sigma(j)) = \sum_\ell a_\ell^q(\sigma(i), \sigma(j))$. Let σ be the permutation obtained in the following way: if

$$P = \begin{pmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_m \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & \cdot & \cdot & \cdot & p_{1n} \\ p_{21} & p_{22} & \cdot & \cdot & \cdot & p_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{m1} & p_{m1} & \cdot & \cdot & \cdot & p_{mm} \end{pmatrix},$$

then for a row vector $p_i = (0, 0, \ldots 1, \ldots, 0)$ is $\sigma(i) = j$, i.e. σ in each $i \in \{1, 2, \ldots m\}$ indicates the place in which the vector p_i has the 1. For instance in

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \sigma(1) = 1, \sigma(2) = 3, \sigma(3) = 2.$$

Therefore it has $B(q)(i, j) = P^{-1}(i, j)A(q)(i, j)P(i, j) = \sum_r \sum_s p_{rs}p_{sj}A(q)(i, j)$ and $B(q)(\sigma(i), \sigma(j)) = \sum_r \sum_s p_{rs}p_{sj}A(q)(\sigma(i), \sigma(j))$ but $p_{rs}p_{sj} = \delta_{i,\sigma(r)}\delta_{j,\sigma(s)}$, thus $(B \circ \sigma)(q) = A(q)$ and so $A \sim_2 B$. □

Finally we arrive to

Theorem 1. *Let $(\mathcal{G}_A, \Sigma_A)$ be a representation of a lattice system with m spins, where the matrix $A(q) \in \mathcal{M}_m(\mathbf{Z}^+[\exp])$ has the property that any non-trivial product of powers of its entries is not equal to* 1. *Then the free energy β_A uniquely determines the matrix A, up to $\sim_2$equivalence.*

Proof: Follows linking the above results. □

3. CLASSIFICATION OF GIBBS STATES ASSOCIATED TO MATRICES BY THE NON-MARKED SPECTRUM

In our approach we are considering matrices as some kind of "observable," and in the introduction we have anticipated a notion of *Gibbs states* associated to a matrix which entries in $\mathcal{M}_m(\mathbf{Z}^+[\exp])$ and the corresponding directed and weighted graphs. Recall that it was done by introducing the n volume—Gibbs ensembles $\mu_{n,A(q)}$ as point mass distributions with mass $w_A^q(\gamma)$, for a path γ of length n, and Gibbs-Boltzmann factor $Z_n(q) = \sum_{|\gamma|=n} w_A^q(\gamma)$. The Gibbs state $\mu_{A(q)}$ is defined as a "thermodynamic limit" of the finite volume ensembles. Recall also that the *cylinder of length n* for a path $\gamma = e_0 \dots e_{n-1}$ is the set

$$C_n = C_n(\gamma) = \{x \in \Sigma_A : x_i = e_i, i = 0, 1, \dots, n-1\}.$$

The space of configurations Σ_A can be partitioned in cylinders: we can consider, see introduction, the configurations identified with infinite paths $\gamma = e_0 e_1 \dots$ in $\mathcal{G}_A$, let us denote by $in(e)$ the initial vertex of e, and if $\Omega = \{0, 1, \dots, m-1\}$ is the numeration of the spins of the system, then let

$$G_i = \{\gamma = e_0 e_1 \dots : \ in(e_0) = i\}, \qquad i = 0, 1, \dots, m-1. \tag{9}$$

Now $\Sigma_A = \bigcup_{i=0}^{m-1} G_i$ (disjoint union).

As we mentioned in the introduction one of the objectives is to prove that the ratio $\frac{\mu_{A(q)}(C_n(\gamma))}{w_A^q(\gamma)\exp(-nF_A(q))}$ is uniformly upper and lower bounded, for any n, q, γ. Before doing this we must to introduce some background. In the space $\Sigma_A = \{x : x = (x_k)_{k \in \mathbf{Z}^+} : A(x_k, x_{k+1}) = 1, \ x_k \in \Omega\}$ can be put the metric $d_t(x, y) = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{t^k}$, $t > 1$. It does not matter which value of t is considered because all metrics d_t induce the same topology,[5] but it is convenient to take a large value of t. The topology induced by the metrics make Σ_A a compact space and agrees with the topology product of discrete topology in Ω. The distance for finite sequences or finite paths can be obtained as induced by the metric d_t taking a finite sum: if $\gamma = e_0 \dots e_{n-1}$, $\gamma' = e'_0 \dots e'_{n-1}$ then $d_t(\gamma, \gamma') = \sum_{k=0}^{n-1} \frac{|in(e_i) - in(e'_i)|}{t^k}$. The following metric can be obtained from d_t, let d_t^n defined in such a way that if $x \in \Sigma_A$, and $\Pi_n(x)$ is the truncation of the infinite sequence x to its first n terms which is represented by the path γ, then the d_t^n–ball centered in x with radius $\varepsilon = t^{-n}/2$ equals the cylinder $C_n(\gamma)$ for any $x \in C_n(\gamma)$, or two points x, y are within δ–distance in d_t^n if and only if all the paths representing their truncations to sequences of length $\leq n$, are within δ–distance in d_t. A set $E \subset \Sigma_A$ is said to be *n–separated* if for any $x, y \in E, x \neq y$, it holds $d_t^n(x, y) > \varepsilon = t^{-n}/2$, it means that all the paths representing $\Pi_i(x)$, $\Pi_i(y)$, $i \leq n$, can be distinguished with precision ε. Due to the compactness of Σ_A the separated sets are finite.

If A is an aperiodic matrix and γ is an admissible path of length n, then $\Sigma_A \cap C_n(\gamma) \neq \emptyset$ and contains a sequence x such that a finite restriction of x gives a cycle.[5] This expresses the density of the cycles in the space of configurations. In particular under the aperiodicity of the matrix the system $(\mathcal{G}_A, \Sigma_A)$ has the so called *specification property*, due to Bowen,[1] which naively states that for specified admissible paths in $\mathcal{G}_A$ can be found a closed path, i.e. a cycle, approximating them with a certain precision. More formally the specification is defined as follows: for any $\delta > 0$ there is an integer $p(\delta)$ such that if $\mathcal{I} = \{n_1, n_2, \ldots, n_k\} \subset [a, b]$ is an interval of positive integers and $x_1, x_2, \ldots, x_r \in \Sigma_A$ then there exists a cycle γ of length $(b - a)+ p(\delta)$ such that $d_t(\gamma_j, \gamma_j^i)$, where γ_j is the restriction of γ to its j first edges and γ_i^j are the paths representing $\Pi_j(x_i)$, $j = n_1, n_2, \ldots, n_k$, $i = 1, 2, \ldots, r$.

Another result to be used later is that the weights as functions on the edges has a "bounded distortion" if consider paths which within a distance not exceeding a certain ε, this means if $\gamma = e_0 \ldots e_{n-1}$, $\gamma' = e'_0..e'_{n-1}$ are admissible paths with $d_t(\gamma, \gamma') < \varepsilon$ then

$$C^{-1} \leq \frac{\prod_{i=0}^{n-1} w_A^q(e_i)}{\prod_{i=0}^{n-1} w_A^q(e'_i)} \leq C,$$

for some constant $C > 0$ and for any positive integer n with a fixed q. This conditions can be rewritten as

$$\left| \sum_{i=0}^{n-1} \log w_A^q(e_i) - \sum_{i=0}^{n-1} \log w_A^q(e'_i) \right| < K,$$

for some K. This can established adapting a result formulated in a more general context (for instance see Ref. 6).

Let us consider a partition function defined from a counting of points in separated sets: let

$$N_n(q) = \sup \left\{ \sum_{\gamma = \Pi_n(x), x \in E} w_A^q(\gamma) : E \text{ is } n\text{–separated} \right\}, \tag{10}$$

by $\Pi_n(x) = \gamma$ we actually mean $\Pi_n(x)$ is represented by the path γ. Then let

$$G_A(q) = \lim_{n \to \infty} \frac{1}{n} \log(N_n(q)). \tag{11}$$

Proposition 3. *The function $G_A(q)$ equals the free energy $F_A(q)$.*

Proof: Let $n \geq p(\delta)$ an let E be a $n - p(\delta)$ separated set ($p(\delta)$ is the specification number), if $x \in E$ then by the specification property there is a n–length cycle γ such that $d_t(\gamma_{n-p(\delta)}, \eta_{n-p(\delta)}) < \delta$, where $\eta_{n-p(\delta)}$ represents $\Pi_{n-p(\delta)}(x)$,recall that with γ_j we denote the restriction of γ to its j first edges. The assignation of γ to any configuration x is injective. Let us denote by e_i the edges of γ and by e'_i denoting the edges of η, we have

$$\sum_{i=0}^{n-1} \log w_A^q(e_i) = \sum_{i=0}^{n-p(\delta)-1} \log w_A^q(e_i) + \sum_{i=0}^{p(\delta)-1} \log w_A^q\big(e_{i+n-p(\delta)}\big)$$

and by the mentioned bounded distortion property of the weights we get

$$\sum_{i=0}^{n-1} \log w_A^q(e_i) \geq \sum_{i=0}^{n-p(\delta)-1} \log w_A^q(e'_i) - K - \sum_{i=0}^{p(\delta)-1} \log w_A^q(e_{i+n-p(\delta)}),$$

let ϕ a map defined on paths which assigns, for fixed q, to any γ the number $\log w_A^q(in(\gamma))$, where $in(\gamma)$ denotes the initial edge of γ, so that

$$\left|\sum_{i=0}^{N-1} \log w_A^q(e_i))\right| \leq N\|\phi\|_0,$$

for any sequence of N edges considered. In this way we can write

$$\sum_{i=0}^{n-1} \log w_A^q(e_i) \geq \sum_{i=0}^{n-p(\delta)-1} \log w_A^q(e'_i) - K - p(\delta)\,\|\phi\|_0\,. \tag{12}$$

Thus summing over the cycles of length n is obtained

$$Z_n(q) = \sum_{|\gamma|=n} w_A^q(\gamma) \geq \exp(-K - p(\delta)\,\|\phi\|_0) \times N_{n-p(\delta)}\,(q)\,, \tag{13}$$

for n enough large.

To prove the opposite inequality, we firstly point out that the space of configurations Σ_A has the following property of expansiveness: for any $x, y \in \Sigma_A$, $x \neq y$, there is constant $\delta > 0$ such that $d_t^n(x, y) > \delta$ for some positive integer n, this means that all paths representing the truncations $\Pi_i(x)$, $\Pi_i(y)$, $i \leq n$, can be distinguished with precision δ. To see that certainly Σ_A possesses this property notice that the partition $G = \{G_i\}$, $G_i = \{\gamma = e_0 e_1 \ldots : in(e_0) = i\}$ is such that if $\{G_{x_k}\}$ is sequence of members of G indexed by elements of $\Omega = \{0, 1, \ldots, m-1\}$ then $\bigcap_{k=0}^{\infty} G_{x_k}$ has an only point which is precisely the configuration $x = (x_k)_{k\in\mathbf{Z}^+}$. Let δ the Lebesgue number of the covering G, recall that Σ_A is compact, then it must be $d_t^n(x, y) > \delta$ for some n, because if it were $d_t^n(x, y) < \delta$ for any n then x, y must belong to a set G_{x_n} for every n and so $x, y \in \bigcap_{k=0}^{\infty} G_{x_k}$, but it is no possible if $x \neq y$. Next we show that the elements of the set $\mathcal{C}_n = \{\gamma : \gamma \text{ is a cycle with } |\gamma| = n\}$ are n–separated with a certain precision, with n enough large. The set $\mathcal{C}_n$ may be also considered as a subset of the space of infinite sequences, i.e. the set Σ_A, indeed we have a natural identification of $\mathcal{C}_n$

with $\{x : \Pi_n(x)$ is represented by a cycle γ with $|\gamma| = n\} \subset \Sigma_A$, this identification is done by extending a finite cycle $\gamma = e_0 e_1 \dots e_{n-1}$, with $e_{n-1} = e_0$, to an infinite sequence by infinitely adding to γ periodic blocks $e_0 e_1 \dots e_{n-2} e_0$. Let now $x, y \in \mathcal{C}_n$ and let γ, γ' be the representatives of $\Pi_n(x)$, $\Pi_n(y)$ respectively and we take

$$\eta := \max\left\{d_t(\gamma_i, \gamma_i') : i = 1, 2, \dots, n\right\},$$

where as ever γ_i, γ_i' mean the restriction to the first $i-$edges. So that for every n the representatives of $\Pi_n(x)$, $\Pi_n(y)$ are within d_t–distance at most η, because the periodicity of the sequences, therefore it should be $\eta > \delta$ (the constant of expansiveness), otherwise it would be $x = y$ by definition of expansiveness. Thus $d_t^n(x, y) > \eta > t^{-n}/2$, for enough big n.

Now, since for calculating $Z_n(q)$ the sum is taken over the cycles in $\mathcal{C}_n$, which are as seen separated and

$$N_n(q) = \sup\left\{\sum_{\gamma = \Pi_n(x), x \in E} w_A^q(\gamma) : E \text{ is } n\text{–separated}\right\}$$

it has $Z_n(q) \le N_n(q)$ for n large. Taking corresponding limits concludes $G_A(q) = F_A(q)$. □

Next we established a key result for the classification of Gibbs states.

Theorem 2. *If $\mu_{A(q)}$ is the Gibbs state associated to a matrix $A(q) \in \mathcal{M}_m(\mathbf{Z}^+[\exp])$ then for any n, q and for any path γ of length n, there are constants $A_1, A_2 > 0$ such that*

$$A_1 \le \frac{\mu_{A(q)}(C_n(\gamma))}{w_A^q(\gamma)\exp(-nF_A(q))} \le A_2 \tag{14}$$

Proof: Let $p(\delta)$ be the number of the specification property and let $r \ge n + 2p(\delta)$, $s = r - n - 2p(\delta)$. Recall that the r–volume ensemble of a cylinder C_n is given by

$$\mu_{r,A(q)}(C_n) = \frac{\sum_{\gamma \in C_n \cap \mathcal{C}_r} w_A^q(\gamma)}{\sum_{\gamma \in \mathcal{C}_r} w_A^q(\gamma)}.$$

Let E_s be a maximal s–separated (in this case maximal means with maximal cardinality among the s–separated sets), if $x \in E_s$ then by the specification property there is a $\gamma \in C_n \cap \mathcal{C}_r$, injectively assigned, such that $d_t((\gamma_{n+p(\delta)})_s, \gamma') < \delta$, where γ' represents $\Pi_s(x)$. Let $y \in \Sigma_A$ with a representative $\eta = f_0 f_1 \dots f_{n-1}$

for $\Pi_n(y)$. Let us denote with e 's the edges of γ and with e' 's the edges of γ'. By the specification condition and the bounded distortion property, we have, by fixing q :

$$\left|\sum_{i=0}^{s-1} \log w_A^q(e_i') - \sum_{i=0}^{s-1} \log w_A^q(e_{i+n+p(\delta)})\right| < K$$

and

$$\left|\sum_{i=0}^{n-1} \log w_A^q(e_i) - \sum_{i=0}^{n-1} \log w_A^q(f_i)\right| < K$$

for some constant K. Therefore

$$\left|\sum_{i=0}^{r-1} \log w_A^q(e_i) - \sum_{i=0}^{s-1} \log w_A^q(e_i') - \sum_{i=0}^{n-1} \log w_A^q(f_i)\right|$$

$$= \left|\sum_{i=0}^{n-1} \log w_A^q(e_i) - \sum_{i=0}^{p(\delta)-1} \log w_A^q(e_{i+n}) - \sum_{i=0}^{n-1} \log w_A^q\left(e_{i+n+p(\delta)}\right)\right.$$

$$\left. - \sum_{i=0}^{s-1} \log w_A^q(e_i') - \sum_{i=0}^{n-1} \log w_A^q(f_i) - \sum_{i=0}^{p(\delta)-1} \log w_A^q\left(e_{i+n+s+p(\delta)}\right)\right|$$

$$\leq 2K + \sum_{i=0}^{p(\delta)-1} \log w_A^q\left(e_{i+n+p(\delta)}\right) + \sum_{i=0}^{p(\delta)-1} \log w_A^q\left(e_{i+n+s+p(\delta)}\right)$$

$$\leq 2(p(\delta)\|\phi\|_0) + 2K.$$

Thus

$$\exp(-2(p(\delta)\,\|\phi\|_0))\exp(-2K) \leq \frac{\prod_{i=0}^{r-1} w_A^q(e_i)}{\prod_{i=0}^{s-1} w_A^q(e_i') \prod_{i=0}^{n-1} w_A^q(f_i)}$$

$$\leq \exp(2(p(\delta)\,\|\phi\|_0))\exp(2K).$$

Then we have

$$\mu_{r,A(q)}(C_n(\eta)) = \frac{\sum\limits_{\gamma\in C_n(\eta)\cap\mathcal{C}_r} w_A^q(\gamma)}{\sum\limits_{\gamma\in\mathcal{C}_r} w_A^q(\gamma)}$$

$$\geq \frac{\exp(-2(p(\delta)\|\phi\|_0))\exp(-2K)\prod\limits_{i=0}^{n-1} w_A^q(f_i)\sum\limits_{\substack{\gamma'=\Pi_s(x)\\ x\in E_s}} w_A^q(e_i')}{\sum\limits_{\gamma\in\mathcal{C}_r} w_A^q(\gamma)},$$

where in the sum, as before, by $\gamma' = \Pi_s(x)$ we mean γ' represents $\Pi_s(x)$. Therefore if

$$N_s(q) = \sup\left\{\sum_{\gamma'=\Pi_n(s),x\in E_s} w_A^q(\gamma) : E_s \text{ is } s - \text{separated}\right\}$$

then

$$\mu_{r,A(q)}(C_n(\eta)) \geq \frac{\exp(-2(p(\delta)\|\phi\|_0))\exp(-2K)\prod_{i=0}^{n-1} w_A^q(f_i)N_s(q)}{\sum_{\gamma\in\mathcal{C}_r} w_A^q(\gamma)},$$

and by the proposition 3 and the definition of $F_A(q)$ we have, for s, r large enough, $N_s(q) \geq L\exp(sF_A(q))$, $\sum_{\gamma\in\mathcal{C}_r} w_A^q(\gamma) \leq M\exp(rF_A(q))$, L, M constant. Thus

$$\mu_{r,A(q)}(C_n(\eta)) \geq \exp(-rF_A(q))\exp(-2(p(\delta)\,\|\phi\|_0))\exp(-2K)\frac{L}{M}\prod_{i=0}^{n-1} w_A^q(f_i)$$

$$\geq K_1\exp(-nF_A(q))\prod_{i=0}^{n-1} w_A^q(f_i), \tag{15}$$

with $K_1 = \exp(-2(p(\delta)\|\phi\|_0))\exp(-2K)\dfrac{L}{M}$. To take the weak limit of $\mu_{r,A(q)}$, can be considered the sequence $n_k = r = n + 2p(\delta)$, and with $k \to \infty$ gets

$$\mu_{A(q)}(C_n(\eta)) \geq K_1\exp(-nF_A(q))\prod_{i=0}^{n-1} w_A^q(f_i). \tag{16}$$

For finding the other bound proceeds in a relatively similar way, so we shall omit some details and just point out the main aspects. Let η be an arbitrary path of length n with edges $f_0 f_1 \ldots f_{n-1}$ and let $\gamma \in C_n(\eta)\cap\mathcal{C}_r$, with edges denoted with $e_0 e_1 \ldots$, we have

$$\left| \sum_{i=0}^{r-1} \log w_A^q(e_i) - \sum_{i=0}^{n-1} \log w_A^q(f_i) - \sum_{i=0}^{r-n-1} \log w_A^q(e_{i+n}) \right| < K.$$

Now

$$\mu_{r,A(q)}(C_n(\eta)) = \frac{\sum_{\gamma \in C_n(\eta)\cap \mathcal{C}_r} w_A^q(\gamma)}{\sum_{\gamma\in\mathcal{C}_r} w_A^q(\gamma)} \leq \frac{1}{\sum_{\gamma\in\mathcal{C}_r} w_A^q(\gamma)} \exp(K) \prod_{i=0}^{n-1} w_A^q(f_i).$$

$$\prod_{i=0}^{r-n-1} w_A^q(e_{i+n}) \leq \frac{\exp(K)}{C_1} \exp(-rF_A(q)) N_{r-n}(q) \prod_{i=0}^{n-1} w_A^q(f_i),$$

for some constant C_1. Then

$$\mu_{r,A(q)}(C_n(\eta)) \leq \frac{\exp(K)}{C_1} C_2 \exp(-nF_A(q)) \prod_{i=0}^{n-1} w_A^q(f_i). \tag{17}$$

And the lower bound is obtained with the weak limit of $\mu_{r,A(q)}$, like above. □

Before establishing the classification theorem we review some material from the Ruelle thermodynamic formalism[11] and basic Ergodic Theory. The *entropy* of a probability measure μ can be calculated (c.f. Shannon–Mc.Millan theorem[9]) as $h(\mu) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(C_n)$, where C_n is any $n-$cylinder. Thus we have by Theorem 2 $\log \beta_A(q) = F_A(q) = h(\mu_{A(q)}) + \lim_{|\gamma|\to\infty} \frac{1}{|\gamma|} \log w_A^q(\gamma)$. The term $\frac{1}{|\gamma|} \log w_A^q(\gamma)$ can be considered as an ergodic average, indeed if we let, for fixed q, the map $\phi_A : \gamma \mapsto \log w_A^q(\gamma) = \sum_{i=0}^{n-1} \log w_A^q(e_i)$, then by the ergodic theorem $\lim_{|\gamma|\to\infty} \frac{1}{|\gamma|} \log w_A^q(\gamma) = \mu(\phi_A)$, $\mu - a.e.$ for every ergodic measure μ. Here is considered the measure as a functional. Therefore:

$$\log \beta_A(q) = F_A(q) = h(\mu_{A(q)}) + \mu_{A(q)}(\phi_A)$$

and so $\mu_{A(q)}$ is an *equilibrium state* for the observable ϕ_A.

The set $I_{\phi_A} = \{\mu : F_A(q) = h(\mu) + \mu(\phi_A)\}$ is a compact convex set whose extremal elements, i.e. those which admit just a trivial convex combination, are the *pure thermodynamic phases.* Let $T_A = \{\mu : F_{A+B}(q) - F_A(q) \geq \mu(\phi_B) :$ for any matrix $B\}$, this set, which is non empty, is called the set of *tangent functionals* to F at A.If the entropy map $\mu \mapsto h(\mu)$ is upper semi-continuous, with the weak topology in the space of measures, then $I_{\phi_A} = T_A$ and the expansiveness property in the space of sequences makes this map upper semi-continuous.[13] So that the equilibrium states are in correspondence with the tangents to the graphics of $F_A(q)$, now for the coexistence of thermodynamic phases the free energy $F_A(q)$, or of course $\beta_A(q)$, should have singularities. In other words a phase transition is detected when the free energy is non differentiable. As we have already mentioned, by a result of Tuncel, for an irreducible matrix A the map $\beta_A(q)$ is analytic, and so in this case there is an unique equilibrium state for the observable ϕ_A for any fixed q. Besides for this observable any equilibrium state is a Gibbs state.

If we let $w_{A+CI}(e) = w_A(e)C$, for any constant C, then we have for any path $\gamma = e_0 \dots e_{n-1}$ that $w_{A+CI}(\gamma) = w_A(\gamma)C^n$. Thus if $A(q)$, $B(q)$ are matrices such that for each n holds $w_A(\gamma) = w_B(\gamma)C^n$, for some C and for any cycle γ of length n then $\mu_{n,A(q)} = \mu_{n,B(q)}$ as is directly seen for the definition of Gibbs states and so that $\mu_{A(q)} = \mu_{B(q)}$. In particular $\mu_{A(q)} = \mu_{A(q)+CI}$. If the reciprocal of this result were proved then it would obtained a classification of Gibbs states. In this vein:

Theorem 3. *If $\mu_{A(q)} = \mu_{B(q)}$ then there is a constant $C = C(q) > 0$ such that $w_A(\gamma) = w_B(\gamma)C^n$, for any n and for any cycle of length n. Or, by above comment, $\mu_{A(q)} = \mu_{B(q)}$ if and only if $\mathcal{S}_A = \mathcal{S}_{B+CI}$.*

Proof: We consider "renormalizations" $\widetilde{A}(q) = A(q) - F_A(q)I$, $\widetilde{B}(q) = B(q) - F_B(q)I$, for which $\mu_{\widetilde{A}(q)} = \mu_{A(q)}$, $\mu_{\widetilde{B}(q)} = \mu_{B(q)}$ and $F_{\widetilde{A}}(q) = F_{\widetilde{B}}(q) = 0$, for every q. Let $\mu_{\widetilde{A}(q)} = \mu_{\widetilde{B}(q)} = \mu$, and $C_n = C_n(\gamma)$ is a n–cylinder, by Theorem 2 there are constants $A_1, A_2 > 0$ such that $A_1 w_{\widetilde{A}}(\gamma) \leq \mu(C_n(\gamma)) \leq A_2 w_{\widetilde{B}}(\gamma)$ and so $w_{\widetilde{A}}(\gamma) \leq \frac{A_2}{A_1} w_{\widetilde{B}}(\gamma)$. If γ is a cycle then is valid $w_{\widetilde{A}}(\gamma) = \lim_{k\to\infty} \frac{1}{k} w_{\widetilde{A}}(\gamma k)$, where γk is the path obtained by juxtaposition to γ the same γ by $k-$times. Thus we have $w_{\widetilde{A}}(\gamma) \leq \lim_{k\to\infty} \frac{1}{k} w_{\widetilde{B}}(\gamma k) = w_{\widetilde{B}}(\gamma)$. By a dual argument the opposite inequality is established. Therefore $w_{\widetilde{A}}(\gamma) = w_{\widetilde{B}}(\gamma)$, so that $w_A(\gamma) = w_B(\gamma)C^n$, with $C = \frac{F_B(q)}{F_A(q)}$. □

4. RIGIDITY FOR LONG RANGE POTENTIALS

As we mentioned in the introduction the rigidity problem for finite interaction can be solved by means of algebraic properties of some matrices. In particular, Pollicott and Weiss considered a free energy for potentials depending on a finite number of coordinates (finite range potentials). The approach we have developed in previous Sections allows to treat more general interactions than Pollicott and Weiss ones (e.g. Markov chains) as we pointed out earlier. Herein our aim is to establish some kind of rigidity results for a class of potentials which include infinite range interactions, i.e. depending on the entire configuration. In this case we restrict ourselves to just interaction potentials.

One interesting example in this situation is the Kac model: let $\Omega = \{\pm 1\}$ where the transition matrix has all entries equal to1 and the potential is $\varphi(x) = Jx_0 \sum_{n=1}^{\infty} x_n \lambda^n$, with $\lambda \in (0, 1)$; $J \in \mathbf{R}$ is a coupling parameter.

In the case of finite range potentials we saw in the Introduction that a primitive matrix can be defined by

$$\mathbf{L} = \mathbf{L}(q)(i, j) = \begin{cases} 0 & \text{if} \quad A(i, j) = 0 \\ \exp(q\varphi(x)) & \text{if} \quad A(i, j) = 1, \end{cases} \tag{18}$$

with $x_0 = i$, $x_1 = j$, for instance in the Ising model $\varphi(x) = Jx_0x_1$ and $\mathbf{L}(i, j) = \exp(Jx_ix_j)$. Taking into account Eq. (3), we saw that, in a case like this, it can be considered as partition function

$$Z_n(q) := Tr\,[\mathbf{L}^n(q)]\,. \tag{19}$$

On the other hand the "thermodynamic limit" $\lim_{n\to\infty} \frac{1}{n} \log Z_n(q)$ does exist and equals $\log E_1(\mathbf{L}(q))$, where E_1 is the leading positive eigenvalue of $\mathbb{L}$.[11] The existence of such a leading eigenvalue is ensured by the Perron–Frobenius theorem, since the matrix is primitive.

If we are in the more general situation of potentials that depend on a infinite number of coordinates, we must work with other class of objects than matrices. They will be *transfer operators,* in the style of those introduced by Ruelle in his thermodynamic formalism.

We start by observing that periodic sequences in the symbolic space

$$\Sigma_A^+ = \left\{x = (x_i)_{i\in\mathbf{N}} : x_i \in \Omega, \forall i \in \mathbf{N}, A(x_i, x_{i+1}) = 1\right\}$$

correspond to infinite paths in the associated graphs $\mathcal{G}$ and the cycles of length $|\gamma| = n$ to sequences with periodic blocks of length n. For any cycle γ we shall write x_γ for the element of Σ_A^+ formed by blocks corresponding to γ.

Let us recall the definition of Bernoulli shifts $\sigma : \Sigma_A^+ \to \Sigma_A^+$ where $(\sigma x)_n = x_{n+1}$. We also consider, for a potential $\varphi \in C(\Sigma_A^+)$, the *statistical sum*

$$S_n(\varphi)(x) = \sum_{i=0}^{n-1} \varphi(\sigma^i(x)) \tag{20}$$

and the partition function

$$Z_n(q) = Z_n(q, \varphi) = \sum_{|\gamma|=n} \exp(S_n(q\varphi)(x_\gamma)). \tag{21}$$

Thus, the free energy, associated to a potential φ will be $F_\varphi(q) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(q)$. As we mentioned for finite range potentials this free energy gives the spectral radius of a matrix $\mathbf{L}(q)$, like in Eq. (18). For the infinite range case we would like to relate $Z_n(q)$ with operators traces. It must be done by considering a special class of potentials that we shall describe below. For a cycle γ a potential $\varphi \in C(\Sigma_A^+)$ the weights are given by $w_\varphi(\gamma) = S_n(\varphi)(x_\gamma)$. Therefore we shall consider the, unmarked, spectrum

$$\mathcal{S}_\varphi = \{(w_\varphi(\gamma), n) : \ \gamma \text{ is a cycle with } |\gamma| = n\}. \tag{22}$$

Next we shall write down the operators needed for our purposes: for $\varphi \in C(\Sigma_A^+)$, let

$$\mathcal{L}_\varphi (\varkappa)(x) = \sum_{i\in\Omega} A(i, \kappa_0) \exp(\varphi(i, x)) \chi((i, x)), \tag{23}$$

where (i, x) is the configuration $(i, x_0, x_{1,\dots})$. The space of finite range potentials, i.e. depending on a finite number of coordinates, is left invariant by $\mathcal{L}$ and so the operator can be reduced in this subspace to a matrix like $\mathbf{L}$ for which the relationship (18) is satisfied.

Let us return to the Kac model, in this case the transfer operator reads:

$$\mathcal{K}_\varphi (\varkappa)(x) = \sum_{i=\pm 1} \exp\left(Jx_0 \sum_{n=1}^{\infty} x_n \lambda^n\right) \chi((i, x)). \tag{24}$$

Next we consider the space of functions $\mathcal{A}_\infty(\Sigma_A^+) := \{\varphi \in C(\Sigma_A^+) : \text{exists a } \chi \in \mathcal{A}_\infty(D_R) \text{ with } \varphi(x) = \chi(\pi(x))\}$, where $D_R = \{z : |z| = R\}$ and π is a projection $\pi : \Sigma_A^+ \to D_R$ defined by the assignation $x \longmapsto \sum_{n=1}^{\infty} x_{n-1}\lambda^n$. The space $\mathcal{A}_\infty(U)$ is that formed by the complex functions holomorphic in U and continuous in $\overline{U}$ (the closure of U), endowed with the norm $\|\chi\| = \sup_{z\in D_R} |\chi(z)|$. On $\mathcal{A}_\infty(\Sigma_A^+)$ the operator $\mathcal{K}_\varphi$ induces another one acting on $\mathcal{A}_\infty(D_R)$, which it shall be denoted also by $\mathcal{K}_\varphi$, in the following way: let $\psi_j : D_R \to D_R$, $\psi_j(z) = \lambda(j + z)$, $j = \pm 1$, and thus

$$\mathcal{K}_\varphi (\varkappa)(z) = \sum_{j=\pm 1} \exp(Jxz) \chi(\psi_j(z)), \tag{25}$$

for $\chi \in \mathcal{A}_\infty(D_R)$.

By using the trace formula deduced in Ref. 7 we have

$$Z_n(q) = (1 - \lambda^n) Tr\left(\mathcal{K}_{q\varphi}^n\right) = Tr\left(\mathcal{K}_{q\varphi}^n\right) - Tr\left(\widetilde{\mathcal{K}}_{q\varphi}^n\right), \text{ with } \widetilde{\mathcal{K}} = \lambda\mathcal{K}, \tag{26}$$

what we were looking for i.e. a relationship in the style of (18) with the operator playing the role of the matrix.

The class of potentials $\varphi : \Sigma_A^+ \to \mathbf{R}$ within we shall work is that for which the following conditions be satisfied:

($C1$) There is a projection $\pi : \Sigma_A^+ \to \mathbf{R}^d$, for some $d \geq 1$, and open sets $\{W_i\} \subset \mathbf{R}^d$ such that
$\pi(\Sigma_A^+) \subset \bigcup_i W_i$ and maps $\psi_i : \bigcup_{j\in\Omega_i} W_j \to W_i$ ($\Omega_j := \{i \in \Omega : A_{i,j} = 1\}$. Besides $\pi(i, x) = \psi_i(\pi(x)) \in \Sigma_A^+$, recall that (i, x) is the configuration $(i, x_0, x_{1,\dots})$.

($C2$) There are neighborhoods $U_i \subset \mathbf{C}^d$ of W_i such that each ψ_i extends holomorphically to $\bigcup_{j\in\Omega_i} U_j$ and applies $\bigcup_{j\in\Omega_i} U_j$ strictly itself. By "strictly inside itself" understands: let D be a bounded connected subspace of a Banach space B and ψ a holomorphic map on D. It says that ψ applies D strictly inside itself if

$$\inf_{z\in D, z'\in B-D} \|\psi(z) - z'\| \geq \delta > 0.$$

($C3$) There exists holomorphic functions φ_i defined on U_i such that $\varphi\,(i, x) = \varphi_i(\psi_i(\pi(x)))$, for any $x \in \Sigma_A^+$.

These conditions allow to define a transfers operators as:

$$\mathcal{L}_\varphi : \bigoplus_{i\in\Omega} \mathcal{A}_\infty(U_i) \to \bigoplus_{i\in\Omega} \mathcal{A}_\infty(U_i)$$

$$(\mathcal{L}_\varphi(\chi))_i(z) = \sum_{j\in\Omega} A(i, j) \exp\left(\varphi_j\left(\psi_j(z)\right)\right) \chi\left(\psi_j(z)\right) \tag{27}$$

A trace formula for such an operator, in the style of the Atiyah-Bott formula on Lefschetz fixed point, is displayed in Ref. 7 as:

$$Tr\left(\mathcal{L}_\varphi\right) = \sum_{i\in\Omega} A(i, i) \exp(\varphi_i(\overline{z}_i)) \frac{1}{\det(1 - D\psi_i(\overline{z}_\kappa))}, \tag{28}$$

where $\overline{z}_i$ is the fixed point of ψ_i and $D\psi$ is the differential map of ψ, seen as a linear operator. It must be pointed out that, by the Earle–Hamilton theorem[2] a map ψ applying strictly a domain D inside itself has exactly a fixed point $\overline{z} \in D$ with $\|D\psi(\overline{z})\| < 1$.

A relevant fact about these transfer operators is that they are *nuclear*. Let us recall that an operator $\mathcal{L}$ acting on a Banach space B is nuclear if there exist sequences $(x_n) \subset B$, $(f_n) \subset B^*$ (the dual space of B) with $\|x_n\| = 1$, $\|f_n\| = 1$ and numbers (ρ_n) with $\sum_{n=0}^{\infty} |\rho_n| < \infty$ such that $\mathcal{L}(x) = \sum_{n=0}^{\infty} \rho_n f_n(x) x_n$ for every $x \in B$. The nuclearity of operators similar to (18) and also for those corresponding to a continuous case was established in Ref. 8. To adapt these demonstrations for operators (18) is immediate and so we will omit it.

Let us consider now the family of operators $\mathcal{L}_q$, which are the transfer operators associated to the family of potentials $\{q\varphi\}$. In this case the condition ($C3$) is formulated as: there exists holomorphic functions $\varphi_{i,q}$ defined on U_i such that $q\varphi\,(i, x) = \varphi_{i,q}(\psi_i(\pi(x)))$, for any $x \in \Sigma_A^+$.

By the Grothendieck theory for nuclear operators[3,4] the *Fredholm determinant* $\det(1 - z\mathcal{L}_q)$ is an entire map in the both two variables z, q and it has the expansion $\det(1 - z\mathcal{L}_q) = \exp(-\sum_{n=1}^{\infty} \frac{z^n}{n} Tr(\mathcal{L}_q^n))$. If the charts ψ_i, defined in $(C1) - (C3)$ are constant then by the Mayer trace formula holds $Z_n(q) := Z_n(q\varphi) = Tr(\mathcal{L}_q^n)$, this is the case for instance of the Ising model and many other statistical systems. If the ψ_i are linear, like in the Kac-model, there is also a relationship between the partition function $Z_n(q)$ and the trace of $\mathcal{L}_q^n$ in the style of (26). The general relationship between partition function and trace is

$$Z_n(q) = \sum_{p=0}^{d} Tr\left[\left(\mathcal{L}_q^{(p)}\right)^n\right], \tag{29}$$

where $\mathcal{L}_q^{(p)}$ are operators defined on $\bigoplus_{\kappa\in\Omega} \bigwedge_p \mathcal{B}(U_\varkappa)$, where $\bigwedge_p \mathcal{B}(U_i)$ is the space of the differential p-forms holomorphic on U_i, as

$$\mathcal{L}_q^{(p)} : \bigoplus_{i\in\Omega} \bigwedge_p \mathcal{B}(U_i) \to \bigoplus_{i\in\Omega} \bigwedge_p \mathcal{B}(U_i), U_i \subset \mathbf{C}^d$$

$$\left(\mathcal{L}_q^{(p)}\left(w_p\right)\right)_i (z) = \sum_{j\in\Omega} A_{i,j} \exp\left(\varphi_{j,q}(z)\right) \bigwedge_p D\psi_j(z)\left(w_p\right)\left(\psi_j(z)\right), \tag{30}$$

here $w_p \in \bigwedge_p \mathcal{B}(Ui)$ and $\bigwedge_p D\psi$ is the p-fold exterior product of differential map $D\psi$ (considered a linear operator). It has $\mathcal{L}_q^{(0)} = \mathcal{L}_q$ and any $\mathcal{L}_q^{(p)}$ is nuclear, this results a natural of extension of the fact that $\mathcal{L}_q^{(0)}$ does. Thus the Fredholm determinant $D_p(z, q) := \det(1 - z\mathcal{L}_q^{(p)})$ is entire in z and q, for any p.

Now for $p = 0$, $d = 1$ and constant charts there is an obvious and direct relationship between the Fredholm determinant and the *Ruelle zeta function*[11] which is defined as

$$\varsigma(z, q) = \varsigma_\varphi(z, q) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(q)\right). \tag{31}$$

We have then $\varsigma(z, q) = \frac{1}{D_0(z,q)}$. If the charts are linear we obtain an expression of the partition function as the difference of $Tr(\mathcal{L}_q^n)$ and a constant by $Tr(\mathcal{L}_q^n)$, like in Eq. (26) for the Kac-model. So that in this case are also related the determinant and zeta. For $d \geq 2$ the connection comes from Eq. (29).

Another result about the transfer operators $\mathcal{L}_q$ is the relationship between the spectral radius $\rho(\mathcal{L}_q)$ and the topological pressure, which is $\rho(\mathcal{L}_q) = \exp(T(q))$. This was proved by Ruelle for the operators (23). In Ref. 8, was established the analyticity of the map $q \longmapsto \rho(\mathcal{L}_q)$, provided condition in the style of $(C1) - (C3)$ were fulfilled, and consequently the absence of phase transitions.

The following proposition will serve to obtain a description of the transfer operators spectrum.

Proposition 4. *The spectrum of the operators $\mathcal{L} = \phi C_\psi$, where C_ψ is the composition operator $C_\psi(\chi)(z) = (\chi \circ \psi)(z)$, acting on space of functions $\mathcal{A}_\infty(U)$ is discrete and is formed by eigenvalues $E_n = \{\phi(\overline{z})(D\psi(\overline{z}))^n\}$ where $\overline{z}$ is a fixed point of ψ and with* 0 *as the unique accumulation point.*

Proof: The fact of that the operators $\mathcal{L} = \phi C_\psi$ have discrete spectrum is actually due to Ref. 7 Let $\psi \in \mathcal{A}_\infty(D)$, we have the eigenvalues equation $\mathcal{L}\chi(z) = \phi(z)\chi(\psi(z)) = E\chi(z)$. Clearly if $\chi(\overline{z}) \neq 0$ then an eigenvalue of $\mathcal{L}$ is $E = \phi(\overline{z})$, where $\overline{z}$ is a fixed point of ψ. If $\chi(\overline{z}) = 0$ then differentiating, with respect to z, the above eigenvalue equation is obtained Ψ

$$D\phi(\overline{z}) \times \chi(\overline{z}) + \phi(\overline{z}) \times D\chi(\overline{z}) D\psi(\overline{z}) = ED\psi(\overline{z}).$$

Thus if $D\phi(\overline{z}) \neq 0$ then $E = \phi(\overline{z})D\psi(\overline{z})$. Now the eigenvalues of $\mathcal{L}$ (recall that it is discrete) is the set

$$E_n = \left\{\phi(\overline{z})(D\psi(\overline{z}))^n\right\}.$$

Recall that by the Earle–Hamilton theorem $\|D\psi(\overline{z})\| < 1$, therefore 0 is the only point of accumulation .

Notice that

$$Tr(\mathcal{L}) = \sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \phi(\overline{z})(D\psi(\overline{z}))^n = \frac{\phi(\overline{z})}{\det(1 - D\psi(\overline{z}))},$$

the Mayer trace formulae. □

Remark. *The above result describes indeed the spectrum of the transfer operators since they are finite sums of composite ones.*

Now we shall show that the Ruelle zeta function determines the equilibrium state for a broader class of potentials than in Ref. 10.

Proposition 5. *It holds $\varsigma_{\varphi_1}(z,q) = \varsigma_{\varphi_2}(z,q) \Longrightarrow \mathcal{S}_{\varphi_1} = \mathcal{S}_{\varphi_2}$ ($\mathcal{S}_{\varphi_1}, \mathcal{S}_{\varphi_2}$ are the unmarked orbit spectra of the potentials φ_1, φ_2 as defined in (22)).*

Proof: We have

$$\varsigma_\varphi(z,q) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(q)\right),$$

with

$$Z_n(q) = \sum_{|\gamma|=n} \exp(S_n(q\varphi)(x_\gamma)).$$

The power expansion determines an analytical function in the disc $|z| < \exp(F_\varphi(q))$. If we have an expression of the form $B(q) = \sum_{i=1}^{N} \lambda_i^q$, $\lambda_i > 0$, then from the Newton identities is deduced that $B(q)$ uniquely determines the λ_i, it just needs to know $B(1)$, $B(2)$,..., $B(N)$. This can be applied to the finite sum $\sum_{|\gamma|=n} [\exp(S_n(\varphi)(x_\gamma))]^q$ and so the terms $S_n(\varphi(x_\gamma))$ are uniquely determined by

$Z_n(q)$. In turn the coefficients $Z_n(q)$ are recovered from the expansion differentiating it with respect to q. In this way the spectrum $\mathcal{S}_\varphi$ is uniquely determined from the zeta function. □

Remark. *In fact the above result can be proved in a more general and abstract context. In can be taken a compact metric space X and a map $f : X \to X$ which satisfies the properties of expansiveness and specification. Here we are restricting to a more Statistical Mechanics point of view, so we present the result in the above level.*

Now we state the main result of this section:

Theorem 4. *For spin lattice systems and potentials φ_1, φ_2 for which the conditions $(C1) - (C3)$ are fulfilled the following rigidity result is verified: $F_{\varphi_1}(q) = F_{\varphi_2}(q) \Longrightarrow \mathcal{S}_{\varphi_1} = \mathcal{S}_{\varphi_2}$, or the free energy determines the unmarked spectrum.*

Proof: The scheme to follow for the demonstration is: firstly we consider the Fredholm determinant $D(z, q)$ and the map $\beta(q) = \frac{1}{\rho(\mathcal{L}_q)} = \exp(-F_\varphi(q))$, so that $D(\beta(q), q) = 0$. Let $P(z)$ be an analytic map such that $P(\beta(q)) = 0$ and with $\beta(q)$ determining P. We show that $P(z)$ is a factor of $D(z, q)$, but we also will prove that is not possible to write $D(z, q) = P(z, q)Q(z, q)$, where P, Q are non-constant maps. So that the Fredholm determinant is in some sense "minimal", and then $\beta(q)$ determines the Fredholm determinant. By the relationship of $D(z, q)$ with the zeta function and by the proposition 3, the claim of the theorem will be proved.

For the above proceed we use an approach based on Tuncel developments.[12] Let

$$\mathcal{R} = \left\{ \sum_{i=0}^{k} n_i a_i^q \ : n_i \in \mathbf{Z}, a_i > 0 \right\},$$

if we set $\exp = \{a^q : a \in \mathbf{R}^+\}$ then $\mathbf{Z}[\exp] = \mathcal{R}$, i.e. $\mathcal{R}$ is the ring of integral combinations of elements in exp, or we can write

$$\mathcal{R} = \left\{ \beta : \mathbf{R} \to \mathbf{R} : \beta(q) = \sum_{i=0}^{k} n_i a_i^q \right\}.$$

If the potential φ depends on a finite number of coordinates, for instance $\varphi = \varphi(x_i, x_j)$, then it can be defined a family of matrices $H(q)$with coefficients in $\mathcal{R} = \mathbf{Z}[\exp]$ by

$$H(q) = \begin{cases} 0 & \text{if} \quad A_{i,j} = 0 \\ \exp^q \varphi(x) & \text{if} \quad A_{i,j} = 1 \end{cases},$$

with $x_0 = i$, $x_1 = j$. If $\beta(q) = \beta_A(q) = \rho(A(q))$, is proved in Ref. 12 that $\beta(q)$is analytic and $\beta_A(1) = \log E_1$, where E_1 is the leading eigenvalue of $A = A(1)$, existing by the Perron–Frobenius theorem.

In our case with the potential depending in general of the entire configuration we shall take $\beta(q) = \frac{1}{\rho(\mathcal{L}_q)} = \exp(-F_\varphi(q))$, which as we point out was proved to be analytic and verifies $D(\beta(q), q) = 0$. Recall that by the Proposition 2 the transfer operators have discrete spectrum and so we can put $D(z, q) = \det(1 - z\mathcal{L}_q) = \prod_{n=1}^{\infty}(1 - zE_n(q))$, where $E_1(q) = \exp(F_\varphi(q))$, so that the z-zeros of the Fredholm determinant are the inverses of the eigenvalues of $\mathcal{L}_q$.

As we anticipate as the beginning of the proof we consider a map $P(z, q)$ with $P(\beta(q), q) = 0$, analytic in z and expanded with coefficients in $\mathcal{R}$. Let $\mathcal{F}$ be field of fractions $\mathcal{R}/\mathcal{R}$ and let $\mathcal{G}$ be the set of expansions of analytic maps with coefficients in $\mathcal{F}$. We consider an ideal $\mathcal{I}$ in $\mathcal{G}$ given by $G \in \mathcal{I}$ if and only if G can be expressed as $G = Q/R$ where $Q = Q(z, q)$is an analytic map in z with expansion with coefficients in $\mathcal{R}$ and $Q(\beta(q), q) = 0$ for some analytic function $\beta(q)$ and $R \in \mathcal{R}$. By the analyticity of $\beta(q)$ the choice does not depend on R. So $\mathcal{I} = \{G : G$ can be expanded with coefficients in $\mathcal{F}$, and $G(\beta(q), q) = 0\}$. Let $\mathcal{I} = P\mathcal{G}$ for some P with coefficients in $\mathcal{F}$, we shall show that the expansion has really coefficients in $\mathcal{R}$. We have that the Fredholm determinant belongs to $\mathcal{I}$ and so it can be written: $D(z, q) = P(z, q)Q(z, q)$, where P and Q have coefficients in $\mathcal{F}$ and D with expansion in $\mathcal{R}$. We then have

$$D = \sum_{n=0}^{\infty} a_n z^n, \text{ with } a_n = \sum_{i_n \in I_n} M_{i_n} A_{i_n}^q \in \mathcal{R},\ I_n \text{ finite}$$

$$P = \sum_{n=0}^{\infty} b_n z^n, \text{ with } b_n = \frac{\sum_{j_n \in J_n} N_{j_n} B_{i_n}^q}{\sum_{j_n \in J_n} N'_{j_n} B_{i_n}^{\prime q}} \in \mathcal{F},\ J_n \text{ finite}$$

$$Q = \sum_{n=0}^{\infty} c_n z^n, \text{ with } c_n = \frac{\sum_{\ell_n \in L_n} U_{\ell_n} C_{\ell_n}^q}{\sum_{\ell_n \in L_n} U'_{i_n} C_{i_n}^{\prime q}} \in \mathcal{F},\ L_n \text{ finite.}$$

For any positive integer n let S_n be the subgroup of $\mathbf{R}^+$ generated by A_{i_n}, $B_{j_n} B'_{j_n}$, C_{ℓ_n}, C'_{i_n} and $\mathbf{Z}[S_n]$ is an unique factorization domain. We have $a_0 + a_1 z + \cdots + a_n z^n = (b_0 + b_1 z + \cdots + a_r z^r)(c_0 + c_1 z + \cdots + c_{n-r} z^{n-r})$, then each b_i can be expressed as $b_i = \widetilde{b}_i/b$ with $\widetilde{b}_i \in \mathbf{Z}[S_n]$ as well as any $c_i = \widetilde{c}_i/c$ with $\widetilde{c}_i \in \mathbf{Z}[S_n]$ and for some b, c such that $(b, \widetilde{b}_1, \ldots, \widetilde{b}_r) = 1, (c, \widetilde{c}_1, \ldots, \widetilde{c}_{n-r}) = 1$. Hence the following expression results an equation in $\mathbf{Z}[S_n]$ $bc(a_0 + a_1 z + \cdots + a_n z^n) = (\widetilde{c}_0 + \widetilde{c}_1 z + \cdots + \widetilde{c}_{n-r} z^{n-r})(\widetilde{b}_0 + \widetilde{b}_1 z + \cdots + \widetilde{b}_r z^r)$, since $\mathbf{Z}[S_n]$ is an unique factorization domain each factor of bc must divide all the $\widetilde{b}_i$ or all the $\widetilde{c}_i$, and besides is invertible. Thus c is a "monomial" and so P has actually coefficients in $\mathcal{R}$. Therefore if $P(z, q)$ has coefficients in $\mathcal{R}$ and $\beta(q)$ is a z-zero of P then this map is a factor of the Fredholm determinant $D(z, q)$.

Next we prove that the Fredholm determinant is minimal. We consider a "truncation"

$$D_N(z,q) := \prod_{n=1}^{N}(1 - zE_n(q)) \in \mathcal{R}[z].$$

In this way

$$D_N(z,q) = 1 + \left(\sum_i E_i\right) z + \left(\sum_{i,j} E_i E_j\right) z^2 + \cdots + \left[(-1)^n \prod_i E_i\right] z^N.$$

Another expression for the Fredholm determinant is

$$D(z,q) = 1 + \sum_{n=1}^{\infty} D_n(q) z^n,$$

where

$$D_n(q) = \sum_{\substack{(i_1,\ldots,i_m)\\ i_1+\cdots+i_m=n}} \frac{(-1)^m}{m!} \prod_{j=1}^{m} \frac{1}{i_j} Tr\left(\mathcal{L}_q^{i_j}\right),$$

so that

$$D_N(z,q) = 1 + Tr(\mathcal{L}_q)z + Tr\left(\mathcal{L}_q^2\right)z + \cdots + \left[\sum_{\substack{(i_1,\ldots,i_m)\\ i_1+\cdots+i_m=n}} \frac{(-1)^m}{m!} \prod_{j=1}^{m} \frac{1}{i_j} Tr\left(\mathcal{L}_q^{i_j}\right)\right] z^N.$$

Let us assume that $D(z,q) = P(z,q)Q(z,q)$, as we seen P, Q have expansions with coefficients in $\mathcal{R}$ if $D(z,q)$ does. We compare the coefficients in each $N-$truncation of D and $P.Q$. Thus

$$\begin{aligned} D_N(z,q) &= 1 + \left(\sum_i E_i\right) z + \left(\sum_{i,j} E_i E_j\right) z^2 + \cdots + \left[(-1)^n \prod_i E_i\right] z^N \\ &= \left[\sum_{j_0 \in J_0} N_{j_0} B_{i_0}^q + \left(\sum_{j_1 \in J_1} N_{j_1} B_{i_1}^q\right) z + \cdots + \left(\sum_{j_r \in J_r} N_{j_r} B_{i_r}^q\right) z^r\right] \\ &\quad \times \left[\sum_{\ell_0 \in L_0} U_{\ell_0} C_{\ell_0}^q + \left(\sum_{\ell_1 \in L_1} U_{\ell_1} C_{\ell_{n1}}^q\right) z + \cdots \right. \\ &\quad \left. + \left(\sum_{\ell_{N-r} \in L_{N-r}} U_{\ell_{N-r}} C_{\ell_{N-r}}^q\right) z^{N-r}\right]. \end{aligned}$$

Notice that the product of the eigenvalues E_i, $i = 1, \ldots, N$ can be considered as the determinant of certain $N \times N-$matrix $H = (a_{i,j})$, so

$$\prod_{i=1}^{N} E_i = \sum_{\sigma \in P_n} a_{1,\sigma(1)} \ldots a_{N,\sigma(N)},$$

where $E_i = E_i(q)$, $a_{i,j} = a_{i,j}(q)$ and P_n is the group of permutations of n-elements. Besides

$$\sum_{i=1}^{N} E_i = Tr(H) = \sum_i a_{i,i}.$$

On the other hand the matrix can be taken H is such that

$$a_{i_1,j_1}^{n_1} \ldots a_{i_k,i_k}^{n_k} \neq 1, \text{ for any } (i_1, \ldots, i_k), \left(i_{j_1}, \ldots, j_k\right) \text{ and } n_{1,\ldots,}n_k \in \mathbf{Z}. \tag{32}$$

The coefficient of z^r in the expansion of $D(z, q)$ is of the form

$$\frac{a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}}{a_{i_1,i_1} \ldots a_{i_r,i_r}},$$

where $\sigma \in P_n$ fixes $(i_1, \ldots, i_r)$, and of z^{N-r} is the form

$$\frac{a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}}{a_{i_1,i_1} \ldots a_{i_{N-r},i_{N-r}}}$$

with $\sigma \in P_n$ fixing $(i_1, \ldots, i_{N-r})$.

Then, we have

$$\sum_{\sigma \in P_n} a_{1,\sigma(1)} \ldots a_{N,\sigma(N)} = \sum_{j_r, \ell_{N-r}} N_{j_r} U_{\ell_{N-r}} B_{j_r}^q C_{\ell_{N-r}}^q,$$

so that there is a correspondence between $a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}$ and the coefficients $B_{i_r}^q C_{\ell_{N-r}}^q$. Thus comparing the coefficients of z^r we have $B_{j_r}^q C_{\ell_0}^q = \frac{a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}}{a_{i_1,i_1} \ldots a_{i_r,i_r}}$ and also a similar expression for z^{N-r}. If $\sigma \in P_n$ does not have fixed points then $a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}$ appears in the constant term of the development of the $D(z, q)$, but is not possible to write it as a product of the coefficients $B_{j_r}^q C_{\ell_{N-r}}^q$. To illustrate this, consider the cyclic permutation $\overline{\sigma} = (1, 2, 3)$ and the sum $\sum_{\sigma \in P_3} a_{1,\sigma(1)} a_{2,\sigma(2)} a_{3,\sigma(3)}$, which of course includes $\overline{\sigma}$. The coefficient of z^2 is a sum of terms $a_{i.j} a_{j,i}$ and $a_{i.i} a_{j,j}$. Now $a_{1,2} a_{2,3} a_{3,4}$ must be of the form $a_{i.j} a_{j,i} a_{m,n}$, which could not be possible by (32). □

We can complete the analysis by setting that the spectrum determines the equilibrium states of the potentials . For this it can be performed a similar approach for the classification of Gibbs states as done in Sec. 3.

ACKNOWLEDGMENTS

Support of this work by Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Universidad Nacional de La Plata and Agencia Nacional de Promoción Científica y Tecnológica of Argentina is greatly appreciated. F.V. is a member of CONICET.

REFERENCES

1. R. E. Bowen, Symbolic dynamics for hyperbolic flows. *Am. J. Math.* **95**:429–460 (1973).
2. C. Earle and R. Hamilton, A fixed point theorem for holomorphic mappings. In: *Global analysis, Proc. Simp. Pure Math.*, vol XIV, S. Chern and S. Smale (Eds.), AMS, Providence, RI (1970).
3. A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires. *Mem. Am. Math. Soc.* **16**, (1955).
4. A. Grothendieck, La théorie de Fredholm. *Bull. Soc. Math. France* **84**:319–384 (1956).
5. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*. Cambridge University Press (1995).
6. J. F. King and J. S. Geronimo, Singularity spectrum for recurrent IFS attractors. *Nonlinearity* **6**:337–348 (1992).
7. D. Mayer, On composition operators on Banach spaces of holomorphic functions. *J. Func. Anal.* **35**:191–206 (1980).
8. A. M. Mesón and F. Vericat, An approach to the problem of phase transition in the continuun. *J. Math. Phys.* **46**:053304/1-5 (2005)
9. W. Parry and P. Walters, *Entropy generators in ergodic theory*. Benjamin, New York (1969).
10. M. Pollicott and H. Weiss, Free energy as a dynamical and geometric invariant (or can you hear the shape of a potential). *Commun. Math. Phys.* **240**:457–482 (2003).
11. D. Ruelle, *Thermodynamic formalism, Enciclopedia of Mathematics*, Addison-Wesley (1978).
12. S. Tuncel, Coefficient rings for beta function classes of Markov chains. *Ergod. Th. Dynam. Sys.* **20**:1477–1493 (2000).
13. P. Walters, *An introduction to ergodic theory*. Springer-Verlag, Berlin (1982).